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Extended matrix Gelfand–Dickey hierarchies: reduction to classical Lie algebras

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Abstract. The Drinfeld–Sokolov reduction method has been used to associate extensions of the matrix r-KdV system with the Lie algebra gl_n. We present reductions of these systems to the fixed point sets of involutive Poisson maps, implementing the reduction of gl_n to classical Lie algebras of type B, C, D. Modifications corresponding, in the first place to factorization of the Lax operator, and then to Wakimoto realizations of the current algebra components of the factorization, are also described.

1. Introduction

We consider examples of constrained KP hierarchies having a Lax operator of the form

$$L = \ell + z_+(\partial + w)^{-1}z_- \qquad \text{with } \ell = \Delta^r\partial^r + u_1\partial^{r-1} + \cdots + u_r \tag{1}$$

where Δ is a constant diagonal matrix with Δ^r having distinct, non-zero entries, $u_0, \ldots, u_r$ are in $\widetilde{gl}_p$, $z_+ \in \widetilde{\mathrm{mat}}(p \times s)$, $z_- \in \widetilde{\mathrm{mat}}(s \times p)$ and w is in $\widetilde{gl}_s$. Here $\mathrm{mat}(m \times n)$ denotes the set of $m \times n$ complex matrices and for any vector space V, $\tilde{V}$ stands for $C^\infty(S^1, V)$. $\mathrm{PDO}(m \times n)$ denotes the set of pseudodifferential operators with coefficients in $\widetilde{\mathrm{mat}}(m \times n)$. We shall use standard splitting $\mathrm{PDO} = \mathrm{PDO}_+ + \mathrm{PDO}_-$ of the space of pseudodifferential operators as a vector space direct sum of differential operators and integration operators. We also use the standard trace-form 'res' on PDO given by $\mathrm{res}\sum a_i\partial^i = a_{-1}$.

We call the systems associated with Lax operators of the form given in (1) *systems of extended Gelfand–Dickey type*. These systems are defined for any integers $r, p \geqslant 1$ and $s \geqslant 0$. For the simplicity of language, we shall formulate our statements keeping in mind the generic case for which $r > 1$ and $s > 0$. Note, however, that all statements are also valid in the special cases for which either $r = 1$ or $s = 0$, even though some of them become trivial. The special cases for which $r > 1$ and $s = 0$ reproduce the standard $p \times p$ matrix Gelfand–Dickey systems. The cases with $r = 1$ correspond to generalized AKNS systems.

Recently, there have been several papers devoted to systems of the above type [1–8]. In [9] it was shown how hierarchies with a Lax operator of the form in (1) can be obtained by

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the Drinfeld–Sokolov (DS) reduction method. Specifically, a graded Heisenberg subalgebra of the loop algebra $gl_n \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ is associated with the partition

$$n = pr + s = \overbrace{r + \cdots + r}^{p \text{ times}} + \overbrace{1 + \cdots + 1}^{s \text{ times}} \tag{2}$$

and a generalized KdV hierarchy having the Lax operator in (1) results from the application of the DS reduction procedure [10] (see also [12–14]) with respect to a grade-one element from this Heisenberg subalgebra, if $r > 1$. In the $r = 1$ special case the DS reduction becomes trivial, but interesting results remain valid.

The Lax operator usually studied in the literature for $s > 0$ is obtained from (1) by choosing $p = 1$ and $w = 0$. In fact, setting $w = 0$ is not advantageous since this Dirac reduction of the phase space leads to non-local Poisson brackets.

In this paper we investigate the discrete symmetries given by involutive Poisson maps on the phase space of an extended Gelfand–Dickey system. Reduction to the fixed point set of such a map yields systems which arise from using the classical Lie algebras B, C, D in the DS approach.

The further purpose of the paper is to study modifications of the above systems. In principle, modification arises via two possible mechanisms. The first can be viewed as an application of the well known factorization approach of Kupershmidt–Wilson [15] (see also [16]). The phase space of the resulting modified system is a direct product whose factors carry linear Poisson structures, typically given by current algebras. The second is a novel construction which involves the so-called Wakimoto realizations of the current algebras, as was described in [17]. We will show that the two mechanisms are in fact closely related.

We shall use the abbreviation 'PB' for Poisson bracket and shall refer to the first and second Adler–Gelfand–Dickey PBs on PDO($p \times p$) [18, 19] as the 'AGD PBs'.

2. Extended Gelfand–Dickey hierarchies

In this section we list the main elements of the theory of systems of extended Gelfand–Dickey type. Many of the results are fully described in [9], whilst at the same time much of this theory is standard and goes back to the work of Adler [18], Gelfand–Dickey [19] and Drinfeld and Sokolov [10].

Let $\mathcal{M}_{\rm DS}$† be the space of quadruples (ℓ, z_+, z_-, w) that appear in (1), i.e. as a space

$$\mathcal{M}_{\rm DS} = (\widetilde{gl}_p)^r \times \widetilde{\rm mat}(p \times s) \times \widetilde{\rm mat}(s \times p) \times \widetilde{gl}_s. \tag{3}$$

The functions on $\mathcal{M}_{\rm DS}$ of interest are local functionals that have the form $H = \int_{S^1} h(u_1, \ldots, u_r, z_+, z_-, w)$ with h being a differential polynomial in the entries of the matrices in its arguments. There are two compatible PBs on $\mathcal{M}_{\rm DS}$, given by the following formulae for the respective Hamiltonian vector fields:

$$\boldsymbol{X}^1_H(\ell) = \left[\ell, \frac{\delta H}{\delta \ell}\right]_+ \qquad \boldsymbol{X}^1_H(z_\pm) = \pm\frac{\delta H}{\delta z_\mp} \qquad \boldsymbol{X}^1_H(w) = 0 \tag{4}$$

† In the Hamiltonian reduction approach [9] $\mathcal{M}_{\rm DS}$ represented a so-called DS gauge: we have kept this nomenclature here.

and

$$\begin{aligned}
\boldsymbol{X}^2_H(\ell) &= \left(\ell\frac{\delta H}{\delta \ell}\right)_+ \ell - \ell\left(\frac{\delta H}{\delta \ell}\ell\right)_+ + \left(\ell\frac{\delta H}{\delta z_-}(\partial+w)^{-1}z_-\right)_+ - \left(z_+(\partial+w)^{-1}\frac{\delta H}{\delta z_+}\ell\right)_+ \\
\boldsymbol{X}^2_H(z_+) &= \operatorname{res}\left(\ell\left(\frac{\delta H}{\delta \ell}z_+ + \frac{\delta H}{\delta z_-}\right)(\partial+w)^{-1}\right) - z_+\frac{\delta H}{\delta w} \\
\boldsymbol{X}^2_H(z_-) &= -\operatorname{res}\left((\partial+w)^{-1}\left(z_-\frac{\delta H}{\delta \ell} + \frac{\delta H}{\delta z_+}\right)\ell\right) + \frac{\delta H}{\delta w}z_- \\
\boldsymbol{X}^2_H(w) &= \frac{\delta H}{\delta z_+}z_+ - z_-\frac{\delta H}{\delta z_-} + \left[\frac{\delta H}{\delta w}, w\right] - \left(\frac{\delta H}{\delta w}\right)'.
\end{aligned} \tag{5}$$

Here the gradients are defined by

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} H(\ell + t\delta\ell, z_\pm + t\delta z_\pm, w + t\delta w) &= \operatorname{Tr}\left(\frac{\delta H}{\delta \ell}\delta\ell\right) \\
&\quad + \int_{S^1}\operatorname{tr}\left(\frac{\delta H}{\delta z_+}\delta z_+ + \frac{\delta H}{\delta z_-}\delta z_- + \frac{\delta H}{\delta w}\delta w\right)
\end{aligned} \tag{6}$$

where Tr stands for $\int \operatorname{tr}\operatorname{res}$ and $\frac{\delta H}{\delta \ell} = \sum_{i=1}^{r}\partial^{i-r-1}\frac{\delta H}{\delta u_i}$. For any $A \in \mathrm{PDO}$, we have the decomposition $A = A_+ + A_-$ defined by the standard splitting of PDO.

The map $\pi : \mathcal{M}_{\mathrm{DS}} \to \mathrm{PDO}(p\times p)$, which assigns the pseudodifferential operator L in (1) to the point (ℓ, z_+, z_-, w) in $\mathcal{M}_{\mathrm{DS}}$, is a Poisson map with respect to the PBs defined by formulae (4) and (5) on $\mathcal{M}_{\mathrm{DS}}$ and the first and second AGD PBs on $\mathrm{PDO}(p\times p)$, respectively. It follows that $M = \pi(\mathcal{M}_{\mathrm{DS}})$—the set of Lax operators of the form (1)—is a Poisson subspace of $\mathrm{PDO}(p\times p)$ with respect to the first and second AGD PBs.

To define the commuting flows on $\mathcal{M}_{\mathrm{DS}}$ we proceed as follows. We first diagonalize the Lax operator $L = \pi(\ell, z_+, z_-, w)$. That is to say we write

$$L = g\hat{L}g^{-1} \tag{7}$$

for g an element of $\mathrm{PDO}(p\times p)$ of the form $g = \mathbf{1}_p + \sum_{k=1}^{\infty} g_k\partial^{-k}$, and we require that $\hat{L}$ is diagonal and g is off-diagonal, which determines them uniquely. For Q a constant, diagonal $p\times p$ matrix, let the functions H_j^Q be defined by

$$\begin{aligned}
H_0^Q(\ell, z_\pm, w) &= \operatorname{Tr}(\hat{L}Q(\Delta\partial)^{-r}) \\
H_j^Q(\ell, z_\pm, w) &= \frac{r}{j}\operatorname{Tr}\left(\hat{L}^{j/r}Q\right) \qquad \text{for } j = 1, 2, \ldots.
\end{aligned} \tag{8}$$

The set of functions H_j^Q for $j = 0, 1, 2, \ldots$ and Q arbitrary yields commuting Hamiltonians on $\mathcal{M}_{\mathrm{DS}}$. The corresponding Hamiltonian vector fields are conveniently expressed in the following form:

$$\begin{aligned}
\boldsymbol{X}^2_{j,Q}(L) &= \boldsymbol{X}^1_{j+r,Q}(L) = [(gQg^{-1}L^{j/r})_+, L] \\
\boldsymbol{X}^2_{j,Q}(z_+) &= \boldsymbol{X}^1_{j+r,Q}(z_+) = \operatorname{res}(gQg^{-1}L^{j/r}z_+(\partial+w)^{-1}) \\
\boldsymbol{X}^2_{j,Q}(z_-) &= \boldsymbol{X}^1_{j+r,Q}(z_-) = -\operatorname{res}((\partial+w)^{-1}z_-gQg^{-1}L^{j/r}) \\
\boldsymbol{X}^2_{j,Q}(w) &= \boldsymbol{X}^1_{j+r,Q}(w) = 0 \qquad \forall j = 0, 1, \ldots.
\end{aligned} \tag{9}$$

These commuting vector fields generate the flows of the extended Gelfand–Dickey hierarchy. If $r = p = 1$ and $s = 0$, then the flows are trivial, and we henceforth exclude this case.

3. Modifications of extended Gelfand–Dickey hierarchies

We next apply a two-step factorization procedure to the Lax operator L that leads to modifications of the flows in (9). By *modification*, we mean that there is a non-invertible Poisson map given in terms of a differential polynomial formula, from the Poisson space of the new (modified) variables to $\mathcal{M}_{\mathrm{DS}}$. The Hamiltonians of the modified flows are the pull-backs of the functions H_j^Q in (8). The first step of the factorization procedure is rather well known [4–6, 9]. The second step was mentioned in passing in [9] but details were omitted. Here we also explain the relationship of this second step to the Wakimoto realizations of the current algebra based on the general linear Lie algebra.

Let us introduce the space $\Theta = (\widetilde{gl}_p)^{r-1} \times \widetilde{gl}_{p+s}$ and endow it with the current algebra PB on each of the components. The points of this space are denoted as $(\theta_1, \ldots, \theta_{r-1}, \theta_r) \in \Theta$. For local functionals F, H on Θ we thus have

$$\{F, H\}(\theta_1, \ldots, \theta_{r-1}, \theta_r) = \sum_{i=1}^{r} \int \mathrm{tr}\left(\theta_i \left[\frac{\delta F}{\delta \theta_i}, \frac{\delta H}{\delta \theta_i}\right] - \frac{\delta F}{\delta \theta_i}\left(\frac{\delta H}{\delta \theta_i}\right)'\right). \tag{10}$$

There is a Poisson map μ from Θ to $\mathcal{M}_{\mathrm{DS}}$ described in [9]. It is important to note that μ is Poisson with respect to the *second* Poisson structure on $\mathcal{M}_{\mathrm{DS}}$ given by (5), and *not* with respect to the first Poisson structure given by (4). We shall not specify μ here, but we give the form of the composition $\Phi = \pi \circ \mu : \Theta \to M \subset \mathrm{PDO}(p \times p)$.

Let us write the matrix $\theta_r \in \widetilde{gl}_{p+s}$ in the form

$$\theta_r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{11}$$

where $a \in \widetilde{gl}_p$, $b \in \widetilde{\mathrm{mat}}(p \times s)$, $c \in \widetilde{\mathrm{mat}}(s \times p)$, $d \in \widetilde{gl}_s$. Fix an integer κ between 0 and $r-1$. Then Φ is given by

$$L = \Delta(\partial + \theta_1)\Delta(\partial + \theta_2) \cdots \Delta(\partial + \theta_\kappa)\Delta[\partial + a - b(\partial + d)^{-1}c]\Delta(\partial + \theta_{\kappa+1}) \cdots \Delta(\partial + \theta_{r-1}). \tag{12}$$

For $\kappa = 0$ there are no factors of the form $(\partial + \theta_i)$ on the left, while for $\kappa = r - 1$ there are none on the right. Of course different choices of κ correspond to different definitions of μ but all of them are related by invertible transformations. Hence, all of the apparently different modifications for the different choices of κ are equivalent.

As the composition of two Poisson maps, Φ is guaranteed to be a Poisson map with respect to the second AGD PB on $\mathrm{PDO}(p \times p)$. A direct proof of the Poisson property of Φ can be obtained using the following results.

Lemma 1. The multiplication map: $\mathrm{PDO} \times \mathrm{PDO} \to \mathrm{PDO}$ defines a Poisson map with respect to the second AGD PB on PDO.

Lemma 2. $\{\partial + \theta | \theta \in \widetilde{gl}_p\} \subset \mathrm{PDO}(p \times p)$ is a Poisson subspace with respect to the second AGD PB, which on this subspace coincides with the current algebra PB appearing in (10) for $i \neq r$.

Lemma 3. The map $\eta : \widetilde{gl}_{p+s} \to \mathrm{PDO}(p \times p)$ defined by

$$\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \partial + a - b(\partial + d)^{-1}c \tag{13}$$

is a Poisson map with respect to the current algebra PB on $\widetilde{gl}_{p+s}$ that occurs in (10) for $i = r$ and the second AGD PB on $\mathrm{PDO}(p \times p)$.

The first two lemmas seem to be part of the general knowledge in the field of integrable hierarchies. Lemma 3 was proved in [9]. The modified flows are defined on the phase space Θ by pulling back the Hamiltonians H_j^Q in (8) by means of the map μ.

For reasons explained in [9] (see also [1, 2]), if $s \neq 0$ the factor

$$K := \partial + a - b(\partial + d)^{-1}c \tag{14}$$

entering the factorization of L in (12) is called the 'AKNS factor'. Then the flows on Θ can themselves be modified by factorizing K as follows

$$K = \partial + a - b(\partial + d)^{-1}c = (\partial + \vartheta_0)(\mathbf{1}_p - \gamma(\partial + \vartheta_1 + \beta\gamma)^{-1}\beta) \tag{15}$$

for

$$(\vartheta_0, \vartheta_1, \beta, \gamma) \in \widetilde{gl}_p \times \widetilde{gl}_s \times \mathcal{S} \qquad \text{where } \mathcal{S} := \widetilde{\text{mat}}(s \times p) \times \widetilde{\text{mat}}(p \times s). \tag{16}$$

We let the space $\widetilde{gl}_p \times \widetilde{gl}_s \times \mathcal{S}$ be endowed with the natural direct sum Poisson structure. That is if F and H are two local functionals on this space, we have

$$\{F, H\}(\vartheta_0, \vartheta_1, \beta, \gamma) = \sum_{i=0,1} \int_{S^1} \text{tr}\left(\vartheta_i \left[\frac{\delta F}{\delta \vartheta_i}, \frac{\delta H}{\delta \vartheta_i}\right] - \frac{\delta F}{\delta \vartheta_i}\left(\frac{\delta H}{\delta \vartheta_i}\right)'\right)$$
$$+ \int_{S^1} \text{tr}\left(\frac{\delta F}{\delta \beta}\frac{\delta H}{\delta \gamma} - \frac{\delta H}{\delta \beta}\frac{\delta F}{\delta \gamma}\right). \tag{17}$$

The factorization specified in (15) can be lifted to a mapping $\nu : \widetilde{gl}_p \times \widetilde{gl}_s \times \mathcal{S} \to \widetilde{gl}_{p+s}$ whose equation is

$$a = \vartheta_0 - \gamma\beta \qquad d = \vartheta_1 + \beta\gamma \qquad b = \vartheta_0\gamma - \gamma\vartheta_1 - \gamma\beta\gamma + \gamma' \qquad c = \beta \tag{18}$$

and direct calculation proves the following.

Proposition 4. If $\widetilde{gl}_p \times \widetilde{gl}_s \times \mathcal{S}$ and $\widetilde{gl}_{p+s}$ are endowed with the PB in (17) and with the current algebra PB, respectively, then the map ν determined by (18) is a Poisson map.

Define the space $\Theta' = \left(\widetilde{gl}_p\right)^{r-1} \times \widetilde{gl}_p \times \widetilde{gl}_s \times \mathcal{S}$ and endow it with the product PB given by the current algebra PB on $\left(\widetilde{gl}_p\right)^{r-1}$ together with the PB in (17). Then ν gives rise to a Poisson map $\nu' : \Theta' \to \Theta$, which acts as ν on $\widetilde{gl}_p \times \widetilde{gl}_s \times \mathcal{S}$ and as the identity on the $\left(\widetilde{gl}_p\right)^{r-1}$ factor. This map provides us with a modification of the system on $\Theta = \left(\widetilde{gl}_p\right)^{r-1} \times \widetilde{gl}_{p+s}$. The resulting modified system is the same as the one engendered by the composite Poisson map $\mu \circ \nu' : \Theta' \to \mathcal{M}_{\text{DS}}$.

We now explain that the map ν defined by (18) can be used to generate a huge family of 'realizations' of the current algebra PB based on gl_m for any m. For this we simply repeat the construction for an arbitrary partition of m of the form $m = m_1 + m_2$. This amounts to writing $\theta \in \widetilde{gl}_m$ as $\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \in \widetilde{gl}_{m_1}$ etc, and expressing a, b, c, d by formula (18) in which we then insert the variables

$$(\vartheta_0, \vartheta_1, \beta, \gamma) \in \widetilde{gl}_{m_1} \times \widetilde{gl}_{m_2} \times \mathcal{S}_{m_1,m_2} \qquad \text{with } \mathcal{S}_{m_1,m_2} := \widetilde{\text{mat}}(m_2 \times m_1) \times \widetilde{\text{mat}}(m_1 \times m_2). \tag{19}$$

The PBs of these new variables defined similarly to (17) then imply the current algebra PB for the variable θ, that is we have a Poisson map as in proposition 4. Repeating the construction iteratively for the current algebra factors, we can associate a Poisson map

$$\nu_{m_1,m_2,\ldots,m_l} : \widetilde{gl}_{m_1} \times \widetilde{gl}_{m_2} \times \cdots \times \widetilde{gl}_{m_l} \times \mathcal{S}_{m_1,m_2,\ldots,m_l} \to \widetilde{gl}_m \tag{20}$$

with any partition $m = m_1 + m_2 + \cdots + m_l$. The procedure gives that as a vector space

$$\mathcal{S}_{m_1,m_2,\ldots,m_l} = \widetilde{\mathbb{C}}^d \times \widetilde{\mathbb{C}}^d \qquad \text{for } 2d = m^2 - m_1^2 - m_2^2 - \cdots - m_l^2 \tag{21}$$

and it carries the corresponding canonical PB. The precise formula of the map in (20) depends on the route whereby the final partition of m is reached through the iterative procedure. However, it was proved in [17] (in a more general context) that the various Poisson maps that follow are all related by invertible Poisson maps. The map in (20) is known as a generalized (classical) Wakimoto realization of the current algebra PB based on gl_m. The standard Wakimoto realization belongs to the partition of m for which all $m_i = 1$. An explicit formula for the Wakimoto realizations was derived in [17] by different methods. Further results and background on Wakimoto realizations can also be found in [17] and references therein.

We can use any of the Wakimoto realizations in (20) to modify any of the current algebra factors that appear in the factorization of L in (12). This yields a large family of modifications of the extended Gelfand–Dickey hierarchies.

4. Discrete reductions

We now search for discrete symmetries of the extended Gelfand–Dickey systems. The compatible PBs on the phase space $\mathcal{M}_{\mathrm{DS}} = \{(\ell, z_+, z_-, w)\}$ are given by

$$\{F, H\}_i^* = \boldsymbol{X}_H^i(F) = \mathrm{Tr}\left(\frac{\delta F}{\delta \ell}\boldsymbol{X}_H^i(\ell)\right) + \int_{S^1} \mathrm{tr}\left(\frac{\delta F}{\delta z_+}\boldsymbol{X}_H^i(z_+) + \frac{\delta F}{\delta z_-}\boldsymbol{X}_H^i(z_-) + \frac{\delta F}{\delta w}\boldsymbol{X}_H^i(w)\right) \tag{22}$$

for arbitrary local functionals F, H on $\mathcal{M}_{\mathrm{DS}}$, where $\boldsymbol{X}_H^i$ are defined by (4), (5) for $i = 1, 2$. Specifically, we look for symmetries given by some involutive map

$$\sigma : \mathcal{M}_{\mathrm{DS}} \to \mathcal{M}_{\mathrm{DS}} \qquad \sigma^2 = \mathrm{id} \tag{23}$$

which leaves the PBs invariant,

$$\{F \circ \sigma, H \circ \sigma\}_i^* = \{F, H\}_i^* \circ \sigma \qquad i = 1, 2. \tag{24}$$

We take the following ansatz for σ. Let $m \in GL_p$ and let $q \in GL_s$, i.e. m and q are constant, invertible, respectively $p \times p$ and $s \times s$ matrices. Define the map $\sigma_{m,q} : \mathcal{M}_{\mathrm{DS}} \to \mathcal{M}_{\mathrm{DS}}$ by

$$\sigma_{m,q} : \begin{pmatrix} \ell \\ z_+ \\ z_- \\ w \end{pmatrix} \mapsto \begin{pmatrix} m\ell^\dagger m^{-1} \\ -mz_-^t q^{-1} \\ qz_+^t m^{-1} \\ -qw^t q^{-1} \end{pmatrix} \tag{25}$$

where $\ell^\dagger$ is given by the standard adjoint operation on PDO$(p \times p)$,

$$\ell^\dagger = (-1)^r \Delta^r \partial^r + \sum_{i=1}^{r} (-1)^{r-i} \partial^{r-i} u_i^t \qquad \text{for } \ell = \Delta^r \partial^r + \sum_{i=1}^{r} u_i \partial^{r-i}. \tag{26}$$

It is not hard to verify that $\sigma_{m,q}$ satisfies (24) whenever it maps the phase space $\mathcal{M}_{\mathrm{DS}}$ to itself, which is ensured by the condition

$$m\Delta^r m^{-1} = (-1)^r \Delta^r. \tag{27a}$$

The involutivity of $\sigma_{m,q}$ leads to the further conditions

$$m^t = \epsilon_m m \qquad \epsilon_m = \pm 1 \qquad q^t = \epsilon_q q \qquad \epsilon_q = \pm 1 \qquad \text{with } \epsilon_m \epsilon_q = -1. \tag{27b}$$

Note that if $\epsilon_m = -1$ then p must be *even* and when $\epsilon_q = -1$ then s must be even. For any natural numbers a and b define the $a \times a$ and $2b \times 2b$ matrices η_a and Ω_{2b} by

$$\eta_a = \sum_{i=1}^{a} e_{i,a+1-i} \qquad \Omega_{2b} = \sum_{i=1}^{b} e_{i,2b+1-i} - \sum_{i=b+1}^{2b} e_{i,2b+1-i} \tag{28a}$$

where the $e_{i,j}$ are elementary matrices of appropriate size having a single non-zero entry 1 at the ij position. Let ξ_a denote an arbitrary $a \times a$ diagonal, invertible matrix subject to

$$\eta_a \xi_a \eta_a = -\xi_a \tag{28b}$$

which means that ξ_a is antisymmetric under transpose with respect to the antidiagonal. Using this notation, we have the following types of solutions for $\sigma_{m,q}$. (The notion of a representative example is justified later in this section.)

- Type C1: $r = 2\rho$ even, $\forall p$, m is diagonal and q is arbitrary with $\epsilon_q = -1$, $s = 2l$ even. Representative example: $\sigma_{\Delta,\Omega_{2l}}$.
- Type C2: $r = (2\rho + 1)$ odd, $p = 2k$ even, Δ is such that $\eta_p \Delta \eta_p = -\Delta$, $m = \xi_p \Omega_p$ and q is arbitrary with $\epsilon_q = -1$, $s = 2l$ even. Representative example: $\sigma_{\Delta\Omega_{2k},\Omega_{2l}}$.
- Type B: $r = (2\rho + 1)$ odd, $p = 2k$ even, Δ is such that $\eta_p \Delta \eta_p = -\Delta$, $m = \xi_p \eta_p$ and q is arbitrary with $\epsilon_q = +1$, $s = 2l + 1$ odd. Representative example: $\sigma_{\Delta\eta_{2k},\eta_{2l+1}}$.
- Type D: $r = (2\rho + 1)$ odd, $p = 2k$ even, Δ is such that $\eta_p \Delta \eta_p = -\Delta$, $m = \xi_p \eta_p$ and q is arbitrary with $\epsilon_q = +1$, $s = 2l$ even. Representative example: $\sigma_{\Delta\eta_{2k},\eta_{2l}}$.

Note that the condition $\eta_p \Delta \eta_p = -\Delta$, which is present except for type C1, requires that Δ must have the form $\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_k, -\Delta_k, \ldots, -\Delta_1)$, where $p = 2k$ and $\Delta_i^r \neq \pm \Delta_j^r \neq 0$ for $i \neq j$ since Δ^r must have distinct, non-zero entries. The Lie algebraic meaning of the notation referring to the various types will be explained below.

Given an involutive symmetry $\sigma = \sigma_{m,q}$, one finds that $\sigma : L \mapsto m L^\dagger m^{-1}$ for the Lax operator L in (1). It is not hard to see that this implies that the set of commuting Hamiltonians defined by equation (8) admits a basis consisting of functions which are invariant or anti-invariant (that change sign) with respect to the action of σ. On account of (24), if $H \circ \sigma = H$ then the Hamiltonian vector fields $\boldsymbol{X}_H^i$ are tangent to the fixed point set $\mathcal{M}_{\mathrm{DS}}^\sigma \subset \mathcal{M}_{\mathrm{DS}}$ of σ. Hence, the flows of a 'discrete-reduced hierarchy' may be defined by restricting the flows generated on $\mathcal{M}_{\mathrm{DS}}$ by the σ-invariant Hamiltonians in (8) to the fixed point set $\mathcal{M}_{\mathrm{DS}}^\sigma$. These flows are bihamiltonian with respect to the restricted Hamiltonians and a naturally induced bihamiltonian structure on $\mathcal{M}_{\mathrm{DS}}^\sigma$. The induced PBs on $\mathcal{M}_{\mathrm{DS}}^\sigma$ are defined by restricting the original PBs of functions of σ-invariant linear combinations of the components of ℓ, z_+, z_-, w—which may be regarded as coordinates on $\mathcal{M}_{\mathrm{DS}}^\sigma$—to $\mathcal{M}_{\mathrm{DS}}^\sigma$. The Lax operator of the discrete-reduced system belongs to

$$M^\sigma = \pi(\mathcal{M}_{\mathrm{DS}}^\sigma) = \{L \in M | L = m L^\dagger m^{-1}\}. \tag{29}$$

For fixed p, r, s and a given symmetry type C1, C2, B or D the various possible choices of m and q defining $\sigma_{m,q}$ are equivalent from the point of view of the discrete reduction. In fact, the fixed point sets corresponding to two different choices are always related by a Poisson map of $\mathcal{M}_{\mathrm{DS}}$ given by

$$(\ell, z_+, z_-, w) \mapsto (\bar{m} \ell \bar{m}^{-1}, \bar{m} z_+ \bar{q}^{-1}, \bar{q} z_- \bar{m}^{-1}, \bar{q} w \bar{q}^{-1}) \tag{30}$$

with some constant matrices $\bar{m} \in GL_p$ and $\bar{q} \in GL_s$. It is in this sense that the examples we gave for the symmetries of various types are *representative examples*.

We mentioned that the extended Gelfand–Dickey system follows from an application of the DS reduction procedure to the Lie algebra gl_n. As explained in particular cases in [20], the above discrete reductions are then induced by the reductions of gl_n to a simple complex Lie algebra $\mathcal{G}$ of B, C or D type. This means that the discrete-reduced systems are associated with graded semisimple elements of minimal positive grade from certain graded Heisenberg subalgebras of $\mathcal{G}\otimes\mathbb{C}[\lambda,\lambda^{-1}]$ by means of DS reduction (see also [12, 13] and the review in [14]). Since the graded Heisenberg subalgebras of $\mathcal{G}\otimes\mathbb{C}[\lambda,\lambda^{-1}]$ are classified [11] by the conjugacy classes [21] in the Weyl group $\boldsymbol{W}(\mathcal{G})$ of $\mathcal{G}$, we can label these generalized KdV hierarchies by the respective conjugacy classes in $\boldsymbol{W}(\mathcal{G})$. The conjugacy classes that occur here can be parametrized (as in [20, 21]) by certain 'signed partitions'. The extended Gelfand–Dickey system itself belongs to the conjugacy class $(r,\dots,r,1,\dots,1)$ of $\boldsymbol{W}(gl_n)$ given by the partition in (2). Using this notation, we find that the above discrete symmetries operate on the generalized KdV systems according to the following reduction rules:

$$\begin{aligned}
&\sigma_{\Delta,\Omega_{2l}} : (\overbrace{2\rho,\dots,2\rho}^{p\ \text{times}},\overbrace{1,\dots,1}^{2l\ \text{times}})\subset \boldsymbol{W}(gl_{2(p\rho+l)}) \Longrightarrow (\overbrace{\bar\rho,\dots,\bar\rho}^{p\ \text{times}},\overbrace{1,\dots,1}^{l\ \text{times}})\subset \boldsymbol{W}(C_{p\rho+l})\\
&\sigma_{\Delta\Omega_{2k},\Omega_{2l}} : (\overbrace{r,\dots,r}^{2k\ \text{times}},\overbrace{1,\dots,1}^{2l\ \text{times}})\subset \boldsymbol{W}(gl_{2(kr+l)}) \Longrightarrow (\overbrace{r,\dots,r}^{k\ \text{times}},\overbrace{1,\dots,1}^{l\ \text{times}})\subset \boldsymbol{W}(C_{kr+l})\\
&\sigma_{\Delta\eta_{2k},\eta_{2l+1}} : (\overbrace{r,\dots,r}^{2k\ \text{times}},\overbrace{1,\dots,1}^{2l+1\ \text{times}})\subset \boldsymbol{W}(gl_{2(kr+l)+1}) \Longrightarrow (\overbrace{r,\dots,r}^{k\ \text{times}},\overbrace{1,\dots,1}^{l\ \text{times}})\subset \boldsymbol{W}(B_{kr+l})\\
&\sigma_{\Delta\eta_{2k},\eta_{2l}} : (\overbrace{r,\dots,r}^{2k\ \text{times}},\overbrace{1,\dots,1}^{2l\ \text{times}})\subset \boldsymbol{W}(gl_{2(kr+l)}) \Longrightarrow (\overbrace{r,\dots,r}^{k\ \text{times}},\overbrace{1,\dots,1}^{l\ \text{times}})\subset \boldsymbol{W}(D_{kr+l})
\end{aligned}\tag{31}$$

where $l\geqslant 0$ is arbitrary and $r=2\rho+1$ is odd. With the aid of case-by-case inspection, this result was established in [20] for $r>1$ and $l=0$. Since the remaining cases can be treated in a similar way, we omit the proof (which simply amounts to diagram chasing).

One may try to lift the discrete symmetries given above to analogous symmetries of the modified systems described in section 3. Considering the modified systems that correspond to L in (12), one needs to find a lifted transformation rule $\hat\sigma:\Theta\to\Theta$ for which $\mu\circ\hat\sigma=\sigma\circ\mu$, where $\mu:\Theta\to\mathcal{M}_{\mathrm{DS}}$ is the generalized Miura map. It is clear that such a *local* map exists if and only if the modification is symmetric in the sense that the same number of $(\partial+\theta)$ factors appears to the left and to the right of the special factor K in (12). This modification is available in cases C2, B and D, for which $r=2\rho+1$ and we have

$$L=\Delta(\partial+\theta_1)\cdots\Delta(\partial+\theta_\rho)\Delta[\partial+a-b(\partial+d)^{-1}c]\Delta(\partial+\theta_{\rho+1})\cdots\Delta(\partial+\theta_{2\rho})\tag{32}$$

by choosing $\kappa=\rho$ in (12). The transformation rule $\theta_i\mapsto\theta_i^{\hat\sigma}$ $(0,1,\dots,2\rho)$ is then not difficult to determine by using the requirement that it must imply $L\mapsto L^\sigma=mL^\dagger m^{-1}$ for L in (32). Of course $\hat\sigma$ is a Poisson map, and the corresponding discrete-reduced hierarchy on the fixed point set $\Theta^{\hat\sigma}\subset\Theta$ provides a modification of the hierarchy on $\mathcal{M}_{\mathrm{DS}}^{\sigma}\subset\mathcal{M}_{\mathrm{DS}}$. We leave it to the reader as an exercise to write down the explicit formula of $\hat\sigma$.

For the discrete symmetry of type C1 with $l>0$, a factorized Lax operator of the symmetric form is only available after performing the second factorization of K according to (15). In this case $r=2\rho$, and by choosing $\kappa=\rho-1$ in (12) (and renaming the variables) we indeed obtain the symmetric factorization

$$L=\Delta(\partial+\theta_1)\cdots\Delta(\partial+\theta_\rho)[\mathbf{1}_p-\gamma(\partial+\vartheta+\beta\gamma)^{-1}\beta]\Delta(\partial+\theta_{\rho+1})\cdots\Delta(\partial+\theta_{2\rho}).\tag{33}$$

The modified variables θ_i $(i=1,\dots,2\rho)$ and ϑ,β,γ now belong to the respective factors of the space

$$\Theta'=\big(\widetilde{gl}_p\big)^{2\rho}\times\widetilde{gl}_{2l}\times\widetilde{\mathrm{mat}}(2l\times p)\times\widetilde{\mathrm{mat}}(p\times 2l).\tag{34}$$

The lifted action of the discrete symmetry on Θ' is easy to determine explicitly using the fact that for L in (33) it must imply $L \mapsto L^{\sigma}$ with $\sigma = \sigma_{\Delta,\Omega_{2l}}$.

5. Concluding remarks

In section 4 we saw that many KdV-type hierarchies that are associated with certain conjugacy classes in the Weyl group $\boldsymbol{W}(\mathcal{G})$ for $\mathcal{G}$ a classical simple Lie algebra by generalized DS reduction [12–14] are also obtained as discrete reductions of extended matrix Gelfand–Dickey hierarchies. Note, however, that not all KdV-type hierarchies based on a classical Lie algebra are discrete reductions of hierarchies associated with gl_n. For example, a pseudodifferential operator model of the KdV system associated with the primitive regular conjugacy class $(\bar{p}, \bar{p})$ in $\boldsymbol{W}(D_{2p})$ by generalized DS reduction is not known [20].

In the DS approach modifications of KdV-type systems usually correspond to gauge transformations from certain 'diagonal-type gauges' parametrized by the modified variables to a 'DS gauge' parametrized by the KdV fields. The map $\mu : \Theta \to \mathcal{M}_{\mathrm{DS}}$ was obtained in [9] by using this method. The modification $\nu' : \Theta' \to \Theta$ mentioned after proposition 4 also permits interpretation as a gauge transformation in the DS approach. Moreover, the specific factorizations of L in (32) and (33) that admit a local lifting of the relevant discrete symmetry have a clear interpretation. Namely, these modifications correspond to gauge sections that are mapped to themselves by the original discrete-symmetry transformation that operates on the first-order matrix differential operator variable used in the DS approach. More details on the way discrete symmetries occur in the DS framework can be found in [20].

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